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On the stationary phase evaluation of path integrals in the coherent states representation

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Abstract. The extremal paths that arise in the stationary phase evaluation of coherent state path integrals do not seem to have a simple physical interpretation, in contrast to the extremal paths that occur in the conventional path integrals. On the other hand, a recently derived semiclassical formula for the coherent state propagator involves a path that is determined in exactly the same manner as the extremal paths of the conventional path integrals. Since both the semiclassical and the stationary phase analyses yield asymptotic ($\hbar \to 0$) approximations, the stationary phase and the semiclassical expressions for the propagator should be identical. We present a simple and direct proof that, in spite of the apparent differences, this is indeed the case. The simplification in the semiclassical formula is due to the utilisation of an appropriate set of canonical variables to describe the classical dynamics.

In order to illustrate the usefulness of the semiclassical formula we present an application to the problem of the degenerate parametric amplifier, which had been treated before by operator ordering and path integral methods. The semiclassical approach has a simple classical interpretation that is absent in the alternative treatments.

1. Introduction

Since the introduction of the coherent states in quantum optics by Glauber (1963), these states have found many applications in various fields of physics and chemistry (Glauber 1963, Heller 1976, Davis and Heller 1979, Weissman and Jortner 1981, Hioe 1974, Ruggiero and Zannetti 1982, Yaffe 1982). Consequently, their properties have been thoroughly investigated (Louisell 1973, Klauder and Sudarshan 1968, Perelomov 1971, 1977, Boon and Zak 1978, Bacry *et al* 1978). Since a coherent state may be regarded as the quantum-mechanical analogue of the classical concept of a phase space point, the coherent states representation seems to be most suitable for semiclassical approximations (Heller 1977b).

Perhaps the most important feature of the path integral concept (Feynman and Hibbs 1965) is the expression of the quantum propagator matrix elements in terms of classical objects like phase space, paths and the classical action function. In many situations only a few paths contribute significantly to the path integral, and in these cases it can be reduced to one or more conventional integrals. These paths, to which we shall refer from now on as 'extremal paths', have a simple physical significance: they turn out to be the classical paths that connect the eigenvalues (considered as classical canonical variables) that correspond to the eigenstates between which the

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propagator is computed (Klauder 1978). This reduction of the path integral to conventional integrals is achieved by the application of the stationary phase method (Klauder 1978) and therefore can be regarded as an asymptotic or semiclassical evaluation of path integrals. In fact many semiclassical approximations, that had been derived directly, can be derived from the path integral formulation in this manner (Rajaraman 1975).

Recently, the path integral in the coherent states representation has been considered (Klauder 1978, 1979, Klauder and Daubechies 1982, Faddeev 1975, Blaziot and Orland 1981). However, an application of the stationary-phase approximation to this integral revealed that its extremal path has a rather bizarre structure; it undergoes a discontinuous jump at its two endpoints (Klauder 1978, 1979). When these two singular points are eliminated, a classical path is obtained which, however, is determined by seemingly artificial boundary conditions (Klauder 1978, 1979). These facts obscure the physical interpretation of the coherent state path integral, and thus hinder its usefulness as both a practical and conceptual tool.

An attempt to derive a direct semiclassical approximation to the coherent state propagator was done by Heller (1977a). Heller's approach was based on Miller's general semiclassical theory (Miller 1974). Recently, Heller's results were reformulated, and a simple semiclassical formula for the coherent state propagator was derived (Weissman 1982). This formula is very similar to the stationary-phase approximants of the conventional path integrals, i.e. it involves an action integral over a classical path that is determined by the same boundary conditions that occur in the determination of the extremal paths of the conventional path integrals. Consequently, the semiclassical formula has a simple physical interpretation.

Both the semiclassical formula and the stationary phase approximant to the path integral are asymptotic ($\hbar \to 0$) evaluations of the coherent state propagator. In view of the uniqueness of asymptotic expansions, these two results, in spite of the apparent differences, should be identical. In order to remove any doubt regarding the validity of the semiclassical or the stationary phase formulae we present below a simple and direct proof of their identity. The simplification in the semiclassical formula occurs due to the use of an appropriate set of canonical variables to describe the classical dynamics (Weissman 1982). In the semiclassical approach it was also possible to derive a simple, explicit expression for the pre-exponential factor, which has not been derived in the path integral approach.

As an example, we apply the semiclassical formula to the computation of the coherent state propagator for the degenerate parametric amplifier. The Hamiltonian of this system is quadratic in the creation and annihilation operators and, therefore, the semiclassical approximation yields an exact result in this case. The problem of the degenerate parametric amplifier has been solved already using operator ordering methods (Yuen 1976) and more recently using the coherent state path integral (Hillery and Zubairy 1982). The semiclassical formula yields the correct result in a simpler and more physically transparent way.

2. The stationary phase and the semiclassical formulae

For the sake of completeness and clarity, we present here, without derivation, the stationary phase approximation to the coherent state path integral (Klauder 1979) and the semiclassical approximation to the coherent state propagator (Weissman 1982).

First, it is necessary to introduce the notation that is used for the coherent states exposition. A coherent state $|q_c, p_c\rangle$ can be defined as (Weissman 1982)

$$\langle q|q_c, p_c\rangle = (\pi\sigma^2)^{-1/4} \exp[-\tfrac{1}{2}(q-q_c)^2\sigma^{-2} + i\hbar^{-1}(qp_c - \tfrac{1}{2}q_c p_c)], \tag{2.1}$$

where σ is a constant with dimensions of length. The state $|q_c, p_c\rangle$ is localised in both the q and p representations and

$$\langle q_c, p_c|q|q_c, p_c\rangle = q_c, \qquad \langle q_c, p_c|p|q_c, p_c\rangle = p_c, \tag{2.2}$$

which explains the physical significance of the labels q_c, p_c. Alternatively, we may use a single complex number α:

$$\alpha = \tfrac{1}{2}\sqrt{2}(q_c\sigma^{-1} + i\sigma\hbar^{-1}p_c) \tag{2.3}$$

to label a coherent state (Louisell 1973). This labelling arises from an alternative definition of the coherent state $|\alpha\rangle$ (Louisell 1973)

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{2.4}$$

where a is the boson destruction operator. In fact, the complex labelling is widely used in coherent state theory (Louisell 1973).

The stationary phase approximation to the coherent state path integral is (Klauder 1979)

$$\langle p'', q''|\mathscr{P}(t'', t')|p', q'\rangle \cong \exp\left\{\tfrac{1}{2}i\hbar^{-1}(q''\bar{p}'' - p''\bar{q}'' + \bar{q}'p' - \bar{p}'q') + i\hbar^{-1}\int_{t'}^{t''} [\tfrac{1}{2}(\bar{p}\dot{\bar{q}} - \dot{\bar{p}}\bar{q}) - H(\bar{p}, \bar{q}, t)]\, dt\right\} \tag{2.5}$$

where $\mathscr{P}(t'', t')$ is the time propagator operator:

$$\mathscr{P}(t'', t') = T\exp\left(-i\hbar^{-1}\int_{t'}^{t''} \hat{H}\, dt\right) \tag{2.6}$$

and $[\bar{q}(t), \bar{p}(t)]$ is the classical path that is determined by Hamilton's equations of motion

$$\dot{\bar{q}} = \partial H/\partial\bar{p}, \qquad \dot{\bar{p}} = -\partial H/\partial\bar{q}, \tag{2.7}$$

and the boundary conditions

$$q'\sigma^{-1} + i\sigma\hbar^{-1}\bar{p}' = q'\sigma^{-1} + i\sigma\hbar^{-1}p'. \tag{2.8}$$

$$\bar{q}''\sigma^{-1} - i\sigma\hbar^{-1}\bar{p}'' = q''\sigma^{-1} - i\sigma\hbar^{-1}p''. \tag{2.9}$$

In the above we have used the notation

$$\bar{q}(t') = \bar{q}', \qquad \bar{q}(t'') = \bar{q}'', \tag{2.10}$$

$$\bar{p}(t') = \bar{p}', \qquad \bar{p}(t'') = \bar{p}''. \tag{2.11}$$

To avoid a misunderstanding, let us note that $\bar{q}'$, $\bar{q}''$, $\bar{p}'$, $\bar{p}''$ are, in general, complex while q', q'', p', p'' are real. As we have mentioned above, the extremal path from which equation (2.5) was derived is discontinuous at its endpoints. These discontinuities give rise to the first term in the exponent on the right-hand side of equation (2.5). The operator $\hat{H}$ in (2.6) is the quantum Hamiltonian operator, and H in (2.7) is the classical Hamiltonian that corresponds to it. In the above we have used Klauder's notation (Klauder 1979). However, while Klauder uses units in which $\hbar = \sigma = 1$, we feel that the little extra labour necessary for the retaining of $\hbar$ and σ is justified.

In the semiclassical formula we represent the classical dynamics in terms of the canonical variables (Q, P), that are derived from the usual variables (q, p) by means of the canonical transformation (Weissman 1982)

$$Q = (\tfrac{1}{2}\hbar/\mathrm{i})^{1/2}(q/\sigma - \mathrm{i}\sigma\hbar^{-1}p), \tag{2.12}$$

$$P = (\tfrac{1}{2}\hbar/\mathrm{i})^{1/2}(q/\sigma + \mathrm{i}\sigma\hbar^{-1}p). \tag{2.13}$$

The variables (Q, P) are referred to as 'coherent variables' (Weissman 1982). The coherent variables differ from usual canonical variables in that they are complex for real q and p.

As we have seen, complex numbers can be used to label coherent states, and in the semiclassical theory of coherent states it is convenient to label the coherent states with the coherent variables Q and P.

The semiclassical approximation for the coherent states propagator is (Weissman 1982)

$$\langle Q_2|\mathscr{P}(t_2, t_1)|P_1\rangle = (\partial^2 F/\partial Q_2\,\partial P_1)^{1/2} \exp[\mathrm{i}\hbar^{-1}F - \tfrac{1}{2}\hbar^{-1}(|Q_2|^2 + |P_1|^2)]. \tag{2.14}$$

In the above equation, F is an F_2 type canonical generating function for the canonical transformation $(Q_2, P_2) \to (Q_1, P_1)$ where (Q_1, P_1) and (Q_2, P_2) are the endpoints of a classical path. This path is determined by Hamilton's equations of motion

$$\dot{Q} = \partial H/\partial P, \qquad \dot{P} = -\partial H/\partial Q, \tag{2.15}$$

and the boundary condition

$$P(t_1) = P_1, \qquad Q(t_2) = Q_2. \tag{2.16}$$

It is known from classical mechanics that the generating function F is given by (Goldstein 1972)

$$F = \int_{t_1}^{t_2} \{P(t)\dot{Q}(t) - H[Q(t), P(t)]\}\,\mathrm{d}t + Q_1P_1, \tag{2.17}$$

where $[Q(t), P(t)]$ is the classical path that is determined by (2.15) and (2.16), and Q_1 stands for $Q(t_1)$.

Strictly speaking, the canonical generating functions contain an undetermined additive constant. The reason for choosing the above particular form for F is discussed by Weissman (1982).

The function $H(Q, P)$ that appears in (2.15) and (2.17) is again the classical Hamiltonian that corresponds to the quantum Hamiltonian $\hat{H}$. Klauder (1979) uses the following definition of H

$$H(q, p) = \langle q, p|\hat{H}|q, p\rangle. \tag{2.18}$$

When $\hat{H}$ is expressed in terms of boson creation and annihilation operators, this procedure calls for the substitutions

$$a \to \tfrac{1}{2}\sqrt{2}(q\sigma^{-1} + \mathrm{i}\sigma\hbar^{-1}p), \tag{2.19}$$

$$a^\dagger \to \tfrac{1}{2}\sqrt{2}(q\sigma^{-1} - \mathrm{i}\sigma\hbar^{-1}p), \tag{2.20}$$

in the normally ordered form of $\hat{H}$ (Louisell 1973). However, there are indications (Weissman, unpublished) that better results can be obtained by following the general rule for the correspondence between classical and quantum quantities, which calls for symmetrisation of products of non-commuting operators (Cohen-Tannoudji *et al* 1977). Thus in using the semiclassical formula, we recommend deriving the classical Hamiltonian by making the substitutions

$$a \to (\hbar/\mathrm{i})^{-1/2} P, \tag{2.21}$$

$$\alpha^{+} \to (\hbar/\mathrm{i})^{-1/2} Q, \tag{2.22}$$

in the symmetrised form of $\hat{H}$. In the classical limit ($\hbar \to 0$) both versions coincide. Also for Hamiltonians that are at most quadratic in a and $a^{\dagger}$, the two different procedures yield classical Hamiltonians that differ only by an additive constant and, therefore, in this case the difference between the two procedures for deriving H from $\hat{H}$ is immaterial. Obviously, the semiclassical and the stationary phase formulae become identical only if the same rule is used to derive the classical Hamiltonian H from $\hat{H}$ in both cases.

3. The identity of the stationary phase and the semiclassical formulae

In order to compare both results, let us put

$$P_1 = (\hbar/2\mathrm{i})^{1/2}(q'\sigma^{-1} + \mathrm{i}\sigma\hbar^{-1}p'), \tag{3.1}$$

$$Q_2 = (\hbar/2\mathrm{i})^{1/2}(q''\sigma^{-1} - \mathrm{i}\sigma\hbar^{-1}p''), \tag{3.2}$$

with q', q'', p', p'' real. Using equations (2.12) and (2.13) we can cast the semiclassical boundary conditions (2.16) in the form

$$q'\sigma^{-1} + \mathrm{i}\sigma\hbar^{-1}p' = q(t_1)\sigma^{-1} + \mathrm{i}\sigma\hbar^{-1}p(t_1), \tag{3.3}$$

$$q''\sigma^{-1} - \mathrm{i}\sigma\hbar^{-1}p'' = q(t_2)\sigma^{-1} - \mathrm{i}\sigma\hbar^{-1}p(t_2). \tag{3.4}$$

These boundary conditions are identical to the boundary conditions of the path occurring in the stationary phase approximation that are expressed in equations (2.8) and (2.9). Since the equations of motion for $q(t)$ and $p(t)$ are identical to those of $\bar{q}(t)$ and $\bar{p}(t)$ (provided, of course, that one uses the same Hamiltonians in both cases), we may conclude that the paths $[q(t), p(t)]$ and $[\bar{q}(t), \bar{p}(t)]$ for a given Q_2 and P_1 (or alternatively the corresponding (q'', p'') and (q', p')) are the same. It remains, therefore, to prove the identity of the terms that occur in the exponentials. We start by expressing F in terms of the $(\bar{q}, \bar{p})$ variables:

$$\begin{aligned} F &= \int_{t_1}^{t_2} (P\dot{Q} - H)\,\mathrm{d}t + P_1 Q_1 \\ &= \frac{\hbar}{2\mathrm{i}} \int_{t_1}^{t_2} \left(\frac{\bar{q}}{\sigma} + \frac{\mathrm{i}\sigma}{\hbar}\bar{p}\right)\left(\frac{\dot{\bar{q}}}{\sigma} - \frac{\mathrm{i}\sigma}{\hbar}\dot{\bar{p}}\right)\mathrm{d}t + \frac{\hbar}{2\mathrm{i}}\left(\frac{\bar{q}'^2}{\sigma^2} + \frac{\sigma^2}{\hbar^2}\bar{p}'^2\right) - \int_{t_1}^{t_2} H\,\mathrm{d}t \\ &= \tfrac{1}{2}\int_{t_1}^{t_2} (\bar{p}\dot{\bar{q}} - \bar{q}\dot{\bar{p}})\,\mathrm{d}t + \frac{\hbar}{4\mathrm{i}}\left(\frac{1}{\sigma^2}(\bar{q}''^2 + \bar{q}'^2) + \frac{\sigma^2}{\hbar^2}(\bar{p}''^2 + \bar{p}'^2)\right) - \int_{t_1}^{t_2} H\,\mathrm{d}t. \end{aligned} \tag{3.5}$$

On the other hand we have

$$|P_1|^2 = \tfrac{1}{2}\hbar(q'^2\sigma^{-2} + \sigma^2\hbar^{-2}p'^2), \tag{3.6}$$

$$|Q_2|^2 = \tfrac{1}{2}\hbar(q''^2\sigma^{-2} + \sigma^2\hbar^{-2}p''^2). \tag{3.7}$$

Combining equations (3.5)–(3.7) we get

$$F+\tfrac{1}{2}\mathrm{i}(|Q_2|^2+|P_1|^2)=\tfrac{1}{2}\int_{t_1}^{t_2}(\bar{p}\dot{q}-\bar{q}\dot{\bar{p}})\,\mathrm{d}t+\tfrac{1}{4}\mathrm{i}\hbar\sigma^{-2}(q'^2-\bar{q}'^2+q''^2-\bar{q}''^2)$$
$$+\tfrac{1}{4}\mathrm{i}\sigma^2\hbar^{-1}(p'^2-\bar{p}'^2+p''^2-\bar{p}''^2)-\int_{t_1}^{t_2}H\,\mathrm{d}t. \tag{3.8}$$

From equations (2.8)–(2.9) we deduce that

$$\sigma^{-2}(q'^2-\bar{q}'^2)+\sigma^2\hbar^{-2}(p'^2-\bar{p}'^2)=2\mathrm{i}\hbar^{-1}(q'\bar{p}'-\bar{q}'p'), \tag{3.9}$$
$$\sigma^{-2}(q''^2-\bar{q}''^2)+\sigma^2\hbar^{-2}(p''^2-\bar{p}''^2)=-2\mathrm{i}\hbar^{-1}(q''\bar{p}''-\bar{q}''p'), \tag{3.10}$$

and from equations (3.8)–(3.10) we obtain

$$F+\tfrac{1}{2}\mathrm{i}(|Q_2|^2+|P_1|^2)=\int_{t_1}^{t_2}[\tfrac{1}{2}(\bar{p}\dot{q}-\bar{q}\dot{\bar{p}})-H]\,\mathrm{d}t+\tfrac{1}{2}(q''\bar{p}''-p''\bar{q}''+\bar{q}'p'-\bar{p}'q'). \tag{3.11}$$

Hence, the term in the exponential in equation (2.5) is equal to $\mathrm{i}\hbar^{-1}[F+\tfrac{1}{2}\mathrm{i}(|Q_2|^2+|P_1|^2)]$ which proves the equivalence of the semiclassical and the stationary phase formulae.

4. The semiclassical approximation to the propagator of the degenerate parametric amplifier

In this section we apply the semiclassical formula presented in § 2 to the degenerate parametric amplifier. The quantum Hamiltonian of this system is (Shubert and Wilhelm 1980)

$$\hat{H}=\hbar\omega a^{+}a+k[\exp(2\mathrm{i}\omega t)a^2+\exp(-2\mathrm{i}\omega t)a^{+2}] \tag{4.1}$$

where k and ω are constants. To derive the classical Hamiltonian we rewrite $\hat{H}$ in a symmetric form, using $[a, a^{+}]=1$:

$$\hat{H}=\tfrac{1}{2}\hbar\omega(a^{+}a+aa^{+})-\tfrac{1}{2}\hbar\omega+k[\exp(2\mathrm{i}\omega t)a^2+\exp(-2\mathrm{i}\omega t)a^{+2}]. \tag{4.1a}$$

The classical Hamiltonian $H(Q,P)$ that corresponds to $\hat{H}$ is obtained by substituting $(\hbar/\mathrm{i})^{-1/2}P$ and $(\hbar/\mathrm{i})^{-1/2}Q$ for a and a^{+} respectively in the symmetrised $\hat{H}$ (equation (4.1(a)), which results in

$$H=\mathrm{i}\{\omega QP+k\hbar^{-1}[\exp(2\mathrm{i}\omega t)Q^2+\exp(-2\mathrm{i}\omega t)P^2]\}-\tfrac{1}{2}\hbar\omega. \tag{4.2}$$

Hamilton's equations of motion which correspond to this Hamiltonian are

$$\dot{Q}=\partial H/\partial P=\mathrm{i}\omega Q+2\mathrm{i}k\hbar^{-1}\exp(2\mathrm{i}\omega t)P, \tag{4.3}$$
$$\dot{P}=-\partial H/\partial Q=-\mathrm{i}\omega P-2\mathrm{i}k\hbar^{-1}\exp(-2\mathrm{i}\omega t)Q. \tag{4.4}$$

To solve these equations we introduce

$$\tilde{Q}=\exp(-\mathrm{i}\omega t)Q, \qquad \tilde{P}=\exp(\mathrm{i}\omega t)P. \tag{4.5}$$

The equations of motion (4.3) and (4.4) in terms of the $\tilde{Q}$, $\tilde{P}$ variables read:

$$\dot{\tilde{Q}}=2\mathrm{i}k\hbar^{-1}\tilde{P}, \tag{4.6}$$
$$\dot{\tilde{P}}=-2\mathrm{i}k\hbar^{-1}\tilde{Q}. \tag{4.7}$$

The general solution of these equations is

$$\tilde{Q} = A \exp(2kt\hbar^{-1}) + B \exp(-2kt\hbar^{-1}), \tag{4.8}$$

$$\tilde{P} = -\mathrm{i}A \exp(2kt\hbar^{-1}) + \mathrm{i}B \exp(-2kt\hbar^{-1}), \tag{4.9}$$

where A and B are complex constants, to be determined by the boundary conditions

$$\exp(-\mathrm{i}\omega t_2)Q_2 = A \exp(2kt_2\hbar^{-1}) + B \exp(-2kt_2\hbar^{-1}), \tag{4.10}$$

$$\exp(\mathrm{i}\omega t_1)P_1 = -\mathrm{i}A \exp(2kt_1\hbar^{-1}) + \mathrm{i}B \exp(-2kt_1\hbar^{-1}). \tag{4.11}$$

Solving these equations for A and B and returning to the Q, P variables we get the required trajectory

$$Q(t) = [\exp(\mathrm{i}\omega t)/c(t_2-t_1)][\exp(-\mathrm{i}\omega t_2)Q_2 c(t-t_1) + \mathrm{i}\exp(\mathrm{i}\omega t_1)P_1 s(t-t_2)], \tag{4.12}$$

$$P(t) = [\exp(-\mathrm{i}\omega t)/c(t_2-t_1)][-\mathrm{i}\exp(-\mathrm{i}\omega t_2)Q_2 s(t-t_1) + \exp(\mathrm{i}\omega t_1)P_1 c(t-t_2)], \tag{4.13}$$

$$c(t) = \cosh(2k\hbar^{-1}t), \qquad s(t) = \sinh(2k\hbar^{-1}t).$$

One can now use equation (2.17) to compute $F(Q_2, P_1)$. However, in this case it is simplest to determine F directly, using the equations

$$\partial F/\partial P_1 = Q_1, \tag{4.14}$$

$$\partial F/\partial Q_2 = P_2. \tag{4.15}$$

From equations (4.12) and (4.13) we have

$$Q_1 = Q(t_1) = [\exp(-\mathrm{i}\omega \Delta t)Q_2 - \mathrm{i}\exp(2\mathrm{i}\omega t_1)s(\Delta t)P_1]/c(\Delta t), \tag{4.16}$$

$$P_2 = P(t_2) = [-\mathrm{i}\exp(2\mathrm{i}\omega t_2)s(\Delta t)Q_2 + \exp(-\mathrm{i}\omega \Delta t)P_1]/c(\Delta t), \tag{4.17}$$

where $\Delta t = t_2 - t_1$.

Substituting now the right-hand side of equation (4.16) for Q_1 in equation (4.14) and solving, we get

$$F = [\exp(-\mathrm{i}\omega \Delta t)Q_2 P_1 - \tfrac{1}{2}\mathrm{i}\exp(2\mathrm{i}\omega t_1)s(\Delta t)P_1^2]/c(\Delta t) + f(Q_2), \tag{4.18}$$

where $f(Q_2)$ is a yet unknown function of Q_2. Using (4.15), (4.16) and (4.18) we get

$$\partial f/\partial Q_2 = -\mathrm{i}\exp(-2\mathrm{i}\omega t_2)s(\Delta t)/c(\Delta t)Q_2, \tag{4.19}$$

so that finally

$$F(Q_2, P_1) = \{\exp(-\mathrm{i}\omega \Delta t)Q_2 P_1 - \tfrac{1}{2}\mathrm{i}s(\Delta t)[\exp(2\mathrm{i}\omega t_1)P_1^2 + \exp(-2\mathrm{i}\omega t_2)Q_2^2]\}/c(\Delta t) + C, \tag{4.20}$$

where C is a (possibly time dependent) constant. To determine C let us note that the particular trajectory that is determined by $Q_2 = P_1 = 0$ vanishes altogether, so that for that particular trajectory we have, by equation (2.17),

$$F(0, 0) = -\int_{t_1}^{t_2} H(0, 0, t)\,\mathrm{d}t = \tfrac{1}{2}\hbar\omega \Delta t. \tag{4.21}$$

Since from (4.20) we have $F(0, 0) = C$ we conclude that

$$C = \tfrac{1}{2}\hbar\omega \Delta t. \tag{4.22}$$

We can now compute the pre-exponential factor

$$[\partial^2 F/\partial Q_2\,\partial P_1]^{1/2} = [c(\Delta t)]^{-1/2}\exp(-\tfrac{1}{2}\mathrm{i}\omega\,\Delta t). \tag{4.23}$$

Using equation (2.22) we obtain the final form of the propagator

$$\begin{aligned}\langle Q_2|\mathscr{P}(t_2,t_1)|P_1\rangle \\ &= [c(\Delta t)]^{-1/2}\exp\{\mathrm{i}\hbar^{-1}\exp(-\mathrm{i}\omega\,\Delta t)Q_2P_1/c(\Delta t)+\tfrac{1}{2}\hbar^{-1}\tanh(2k\,\Delta t\hbar^{-1}) \\ &\quad\times[\exp(2\mathrm{i}\omega t_1)P_1^2+\exp(-2\mathrm{i}\omega t_2)Q_2^2]-\tfrac{1}{2}\hbar^{-1}(|Q_2|^2+|P_1|^2)\}. \end{aligned}\tag{4.24}$$

This result coincides with the one obtained before by path integral methods (Hillery and Zubairy 1982). Let us note that had we not used the symmetrised form of $\hat{H}$ to derive H we would have obtained an extra term $\frac{1}{2}\omega\,\Delta t$ in the exponent.

5. Conclusion

We have shown that the semiclassical approximation to the coherent state propagator is just a reformulation of the stationary phase approximant to the coherent state path integral. This reformulation, however, has the merit of providing a simple classical interpretation, which makes the semiclassical formula physically appealing. Our result also suggests that by using the coherent variables, the coherent state path integral can be put in a form which is very similar to the form of the conventional path integrals.

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† We regret the omission of a reference to Klauder's work (1979) in this paper.